Conclusions

This work has demonstrated the feasibility of the V-line as a surface-wave guiding structure supporting higher-order hybrid modes of propagation. Various characteristics of such modes have been discussed qualitatively on the basis of algebraic solutions of the characteristic equation and, more quantitatively, on the basis of numerical solutions obtained for a variety of orders and dielectric constants.

Experimental confirmation of the theory has been successful qualitatively and quantitatively for the second-order principal mode. Verification has been obtained for the exclusion of the transverse electric mode from the V-line with insulated plates, thereby confirming that it is the high-order hybrid modes rather than the simpler low-order transverse modes that are of interest for operation with disjoint plates. The evitability of joining the plates affords distinct advantages in launching and detecting the modes and in the versatility of the V-line.

The V-line configuration with separated plates or truncated apex appears well suited to convenient electronic control of propagation characteristics through the use of a ferroelectric binding medium. The plates serve as supports for the structure, as image surfaces for the guided wave, and as electrodes for the application of bias potentials. The V-line is distinguished from some other configurations that permit bias fields across a binding medium in that it confines the propagating or radiating fields to one side of the dielectric. The region of the apex remains available for auxiliary structures, such as exciting or detecting mechanisms.

Wave Propagation in a Medium with a Progressive Sinusoidal Disturbance*

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Summary—**A recent paper by Simon derives approximate results, employing only three space harmonics, for the propagation characteristics of an electromagnetic wave traveling in a medium possessing a progressive sinusoidal disturbance. A rigorous result is presented here for this same problem, taking into account all of the space harmonics; also, a sufficiency condition for the convergence of this solution is discussed. This sufficiency condition is not satisfied in a particular case treated by Simon. It is shown that his analysis of this case is in error, and that the total field is singular there. The singular nature of the field is associated with "supersonic" effects in the medium containing the progressive disturbance.**

Introduction

A STIMULATING, recent paper[1] by Simon presents solutions for the propagation characteristics of an electromagnetic wave traveling in a medium possessing a progressive sinusoidal disturbance. This disturbance is expressed in terms of a time-varying dielectric constant, in the form

$$\epsilon = \epsilon_0 + \epsilon_1 \cos(\omega_1 t - k_1 z), \quad (1)$$

using the notation employed by Simon. The electromagnetic field is then expanded in terms of spatial harmonics and a relation is found between the amplitudes of these space harmonics. This relation is essentially a system of infinite homogeneous equations with an infinite number of unknowns. Simon then points out that in problems of interest ϵ_1 is very small, and that a rigorous solution to this system of equations is not simple to obtain. He, therefore, adopts a perturbation approach, and retains only the lowest three of the infinite number of space harmonics. With this approximation, he obtains a determinantal equation for the propagation constants, solves this equation for several interesting special cases, and then obtains the corresponding space-harmonic amplitudes.

A major contribution of Simon's paper lies in the stress he places on the interrelation between physical concepts in different disciplines. For example, while it has long been known that a stop band for electromagnetic waves in a periodic structure corresponds to Bragg reflection in crystals, Simon relates the Doppler effect produced in a stop band associated with a moving disturbance to parametric amplification phenomena. The conditions for both up-conversion and down-conversion are considered in some detail, and approximate expressions are presented for the propagation constants and the fields. An additional so-called "triple root" case is also treated in some detail, but the results are of questionable value, for reasons presented below.

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[1] J. C. Simon, "Action of a progressive disturbance on a guided electromagnetic wave," IRE Trans. on Microwave Theory and Techniques, vol. MTT-8, pp. 18–29; January, 1960.

In this note we wish to:

1) present a *rigorous* result to the problem treated by Simon, taking into account *all* of the space harmonics, rather than only three,
2) present a "*sufficiency condition*" between the various physical quantities which, if satisfied, insures the validity of any space-harmonic type of solution, and
3) apply this "sufficiency condition" to the various cases considered by Simon, and indicate the ranges of parameter values to be excluded from his solutions. Particular stress is laid on his triple root case, which does *not* satisfy this condition. Both physical and mathematical reasoning is presented in verification.

Rigorous Result

Over most of the range of frequency and wavenumber values, the perturbation solution of Simon is completely adequate for very small values of ϵ_1. In particular regions, however, the perturbation solution is invalid. For example, as pointed out by Simon, his solution cannot apply to regions corresponding to the higher-order stop bands, or Bragg reflections. It is desirable, therefore, to have available a rigorous solution.

Let us first review, for the sake of clarity, the basic steps involved in the development of such a rigorous solution. We are concerned with the propagation of electromagnetic waves in an infinite medium possessing a fixed permeability μ_0 and a time-varying dielectric constant $\epsilon(z, t)$ of the form given by (1). Such a medium is created by the presence of a traveling disturbance; examples of the source of such a disturbance could be an electromagnetic "pump" wave or an acoustic wave, the latter constituting an example of microwave-phonon interaction. The assumption (1) regarding the properties of the medium thus linearizes the basically nonlinear interaction problem.

Let us restrict the development below to TEM mode propagation in a direction parallel or antiparallel to the moving disturbance. (Simon initially considers a slightly more general case, but he immediately thereafter reduces his relations to those for TEM modes.) Under these conditions, the wave equation for the electric field $E(z, t)$ becomes

$$\frac{\partial^2 E(z,t)}{\partial z^2} = \mu_0 \frac{\partial^2 D(z,t)}{\partial t^2}, \tag{2}$$

with

$$D = \epsilon E, \tag{3}$$

and ϵ given by (1). A representation of the electric field in the disturbed medium is then taken in the form

$$E(z, t) = E_0 e^{-j(\omega t - kz)} P(\omega_1 t - k_1 z), \tag{4}$$

where ω is the angular frequency of the applied signal, k is the macroscopic wavenumber of the wave propagating in the z direction, ω_1 and k_1 are, respectively, the "pump" angular frequency and propagation wavenumber, and P represents a periodic function of its argument with a period of 2π. The notation employed in (4) follows that of Simon. The objective in this development is to obtain a rigorous dispersion relation expressing k as a function of ω and the parameters of the medium.

The right-hand side of (4) is essentially a Floquet representation of the electric field in the moving coordinate system. Let us next expand P into a Fourier series, *i.e.*,

$$P(\omega_1 t - k_1 z) = \sum_{n=-\infty}^{\infty} a_n e^{-jn(\omega_1 t - k_1 z)}. \tag{5}$$

When the series (5) is substituted, along with (4), (1) and (3), into (2), one obtains the following three-term recursion formula for the desired propagation wavenumber k and the unknown electric-field space-harmonic amplitudes a_n:

$$\frac{\epsilon_1}{2\epsilon_0} a_{n+1} + D_n a_n + \frac{\epsilon_1}{2\epsilon_0} a_{n-1} = 0, \tag{6}$$

where

$$D_n = 1 - \left(\frac{\omega}{k_0}\right)^2 \left(\frac{k + nk_1}{\omega + n\omega_1}\right)^2, \tag{7}$$

$$n = 0, \pm 1, \pm 2, \cdots; \qquad k_0 = \omega\sqrt{\epsilon_0 \mu_0}.$$

Relations (6) and (7) are identical with Simon's relation (10), except for his π^2/b^2 term which he drops soon after. Simon then solves (6) and (7) by a perturbation technique which includes only three space harmonics. We present below a rigorous solution which takes into account all of the space harmonics. Relations (6) and (7) are also given by Slater[2] as (20); while Slater then also proposes a perturbation approach employing only three space harmonics, $n=0, \pm 1$, he does not continue with the detailed analysis and interpretation presented by Simon.

In a recent paper,[3] the writers have treated, by a technique commonly employed in the solution of Mathieu-type equations, an infinite set of equations very similar to (6) but with a different expression for D_n. A rigorous solution was obtained in the form of a rapidly convergent continued fraction. Following the

[2] J. C. Slater, "Interaction of waves in crystals," *Rev. Mod. Phys.*, vol. 30, p. 203; January, 1958.

[3] A. A. Oliner and A. Hessel, "Guided waves on sinusoidally-modulated reactance surfaces," IRE Trans. on Antennas and Propagation, vol. AP-7, pp. S201–S208; December, 1959.

derivation presented there, we obtain from (6) the equation:

$$D_0 - \left(\frac{\epsilon_1}{2\epsilon_0}\right)^2 \left[\cfrac{1}{D_1 - \cfrac{(\epsilon_1/2\epsilon_0)^2}{D_2 - \cfrac{(\epsilon_1/2\epsilon_0)^2}{D_3 - \cdots}}} + \cfrac{1}{D_{-1} - \cfrac{(\epsilon_1/2\epsilon_0)^2}{D_{-2} - \cfrac{(\epsilon_1/2\epsilon_0)^2}{D_{-3} - \cdots}}} \right] = 0 \tag{8}$$

for the propagation wavenumber k. The small parameter $\epsilon_1/2\epsilon_0$ is explicitly exhibited in (8); D_n is defined by (7). When (8) is taken to the first order only, it is identical with Simon's relation (15). Expression (8) is rigorous and is rapidly convergent almost everywhere, so that k can be computed to any desired degree of accuracy.

The corresponding electric-field amplitudes can be shown to be given rigorously by:

$$\frac{a_n}{a_{n-1}} = \cfrac{-(\epsilon_1/2\epsilon_0)}{D_n - \cfrac{(\epsilon_1/2\epsilon_0)^2}{D_{n+1} - \cfrac{(\epsilon_1/2\epsilon_0)^2}{D_{n+2} - \cdots}}}, \quad n \geq 1 \tag{9}$$

$$\frac{a_n}{a_{n+1}} = \cfrac{-(\epsilon_1/2\epsilon_0)}{D_n - \cfrac{(\epsilon_1/2\epsilon_0)^2}{D_{n-1} - \cfrac{(\epsilon_1/2\epsilon_0)^2}{D_{n-2} - \cdots}}}, \quad n \leq -1, \tag{10}$$

and

$$\frac{a_n}{a_0} = \frac{a_n}{a_{n-1}} \cdot \frac{a_{n-1}}{a_{n-2}} \cdots \frac{a_1}{a_0}, \quad n \geq 1 \tag{11}$$

$$\frac{a_n}{a_0} = \frac{a_n}{a_{n+1}} \cdot \frac{a_{n+1}}{a_{n+2}} \cdots \frac{a_{-1}}{a_0}, \quad n \leq -1. \tag{12}$$

Hence, after k is found from (8), the D_n values are known and the amplitude ratios, relative to a_0, say, are obtained from relations (9)–(12). To the first order in $\epsilon_1/2\epsilon_0$, (9) and (10) reduce to

$$\frac{a_1}{a_0} = -\frac{\epsilon_1}{2\epsilon_0} \frac{1}{D_1} \tag{13}$$

$$\frac{a_{-1}}{a_0} = -\frac{\epsilon_1}{2\epsilon_0} \frac{1}{D_{-1}}, \tag{14}$$

in agreement with (14) and (12), respectively, of Simon.

Sufficiency Condition for a Space-Harmonic Form of Solution

It can be shown[4] from the theory of three-term recursion formulas, that if the condition

$$|D_n| > \frac{\epsilon_1}{\epsilon_0} \tag{15}$$

is satisfied for all $n > N$, then relations (8) and (9)–(12) converge absolutely and uniformly. A space-harmonic representation of the solution will also converge provided that (15) is satisfied. If (15) is not satisfied, the convergence is not assured, and a perturbation solution in such a range is, at the very least, highly suspect.

A physical interpretation of condition (15) is obtained by considering it in the limit of high n. Thus, as $n \to \infty$,

$$|D_\infty| = \left|1 - \left(\frac{k_1 \omega}{k_0 \omega_1}\right)^2\right| = \left|1 - \left(\frac{v_0}{v_1}\right)^2\right|, \tag{16}$$

where $v_1 = \omega_1/k_1$ and $v_0 = \omega/k_0$ are the phase velocities of the progressive disturbance (pump wave) and the unperturbed electromagnetic wave, respectively. Condition (15) then becomes

$$\left|1 - \left(\frac{v_0}{v_1}\right)^2\right| > \frac{\epsilon_1}{\epsilon_0}, \tag{17}$$

and is satisfied except when the two velocities are almost equal. Simon's triple-root case corresponds exactly to the equality of the two velocities, and thus violates (17). This case is discussed further below.

Double-Root Cases

Simon's double-root cases correspond physically to stop bands, Bragg reflection situations, or parametric conversion regions, depending on the point of view adopted. Only the two lowest stop bands are considered, differing in that for one band the progressive disturbance and the electromagnetic wave propagate in opposite directions, while for the other they travel in the same direction, corresponding to parametric up-conversion and down-conversion, respectively.

[4] J. Meixner and F. W. Schäfke, "Mathieusche Funktionen und Sphäroidfunktionen," Springer Verlag, Berlin, Germany, pp. 89–93; 1954.

For the up-conversion case, the following relation holds:

$$\frac{k_1}{k_0} = -2 - \frac{\omega_1}{\omega} \cdot \tag{18}$$

Substituting (18) into the sufficiency condition (15) for $n \to \infty$, one obtains

$$4\frac{\omega}{\omega_1}\left(\frac{\omega}{\omega_1}+1\right) > \frac{\epsilon_1}{\epsilon_0}, \tag{19}$$

indicating that one should avoid values of ω/ω_1 that are too small. This conclusion is also borne out in Simon's (18), which loses its meaning when ω/ω_1 becomes extremely small.

For the down-conversion case, one finds the relation

$$\frac{k_1}{k_0} = 2 - \frac{\omega_1}{\omega}, \tag{20}$$

so that condition (15) becomes

$$4\frac{\omega}{\omega_1}\left|\frac{\omega}{\omega_1}-1\right| > \frac{\epsilon_1}{\epsilon_0} \cdot \tag{21}$$

As above, one should avoid values of ω/ω_1 that are too small but now, in addition, a region around $\omega = \omega_1$. These results are again in accord with Simon's (29)–(32) which become meaningless in these extremes. The $\omega = \omega_1$ region may also be seen via (20) to be a special case of the triple-root case, which is considered below.

The application of condition (15) to the double-root cases bears out the validity of the solutions to these cases. The same considerations apply to the higher-order stop bands, about which we can briefly comment in view of the availability of the rigorous solutions (8)–(12). The mth-order stop band occurs when D_m becomes small, and the solution is characterized by the presence of two principal space harmonics whose amplitudes a_0 and a_m are both of the same order. Upon inspection of (9) and (11), one sees that

$$\frac{a_m}{a_0} \approx \frac{(-1)^m(\epsilon_1/2\epsilon_0)^m}{D_m \cdot D_{m-1} \cdot D_{m-2} \cdots D_1}, \tag{22}$$

since all values of D_n except that for $n = m$ are not small. Outside of the stop band, the ratio a_m/a_0 is small; it becomes of the order of unity only when D_m becomes sufficiently small. One can see from (9) that this occurs only over a limited range of ω; one may conclude from this that the width of the stop band in ω is proportional to $(\epsilon_1/2\epsilon_0)^m$, and that for high values of m the stop bands become very narrow.

It may be added that Epsztein has independently recognized the parametric up-conversion behavior associated with such stop bands and has proposed[5] a device for producing millimeter waves based on this principle. He has suggested the use of a higher-order stop band so that the frequency conversion ratio may be large, but this introduces practical difficulties because of the very narrow stop band that would be present.

[5] B. Epsztein, "Millimeter Waves," Microwave Res. Inst., Polytechnic Inst. of Brooklyn, N. Y., Rept. No. 840-60, pp. 8-9; July 25, 1960.

Triple-Root Case

Simon's perturbation solution assumes the existence of only three space harmonics, and his triple-root case corresponds to that condition for which all three of his space harmonics must be considered, since none of them is small in amplitude. If the complete solution involving an infinite number of space harmonics is considered, it is seen that the triple-root condition

$$\frac{k_1}{\omega_1} = \frac{k_0}{\omega} \tag{23}$$

implies that none of the infinite number of space harmonics is small in amplitude, and that therefore *all* of them must be considered. This statement becomes clear by considering (6) for any n, and realizing that condition (23) results in *every* D_n being small when solutions in the range k near k_0 are examined. Such a space-harmonic expansion may thus result in a divergent total field. Any solution that includes only three of the space harmonics, and ignores the non-neglectable infinite remainder, must evidently be highly suspect.

From (16) and (17) it is seen that relation (23) clearly violates the sufficiency condition for a space-harmonic form of solution, and that, as Simon recognizes (23) corresponds to equal propagation velocities for the unperturbed electromagnetic wave and the progressive disturbance. As is shown in the Appendix, this effect is analogous to that of a material body passing through the sonic barrier.

The singular nature of the total field is also brought out by considering the asymptotic behavior of the space-harmonic amplitudes. In the range for which (17) is violated, (6) becomes for sufficiently high n

$$\frac{\epsilon_1}{2\epsilon_0} a_{n+1} + D_\infty a_n + \frac{\epsilon_1}{2\epsilon_0} a_{n-1} \cong 0, \tag{24}$$

so that

$$a_n \sim (-1)^n e^{\pm jn\theta}, \tag{25}$$

where

$$\cos\theta = \frac{\epsilon_0}{\epsilon_1} D_\infty, \tag{26}$$

and θ is *real*. Thus, the sum of the squares of the absolute values of the coefficients a_n need not converge. More detailed considerations are presented in the Appendix, where it is shown that a space-harmonic form of expansion will diverge in this range for real values of k, all of which are solutions of (8) or an equivalent determinantal equation.

Further physical and mathematical insight into the singular nature of the solution is afforded by considering certain properties of the original differential equation. In the Appendix, it is shown via a transformation to a moving coordinate system that a singularity in the differential equation occurs whenever the phase velocity v_1 of the moving disturbance is equal to the *local* phase velocity v_L of the electromagnetic wave in the disturbed medium. This equality, which is thus a condition for the appearance of a singularity in the equation, is exactly the complement of (17), which is the sufficiency condition for the solution to be convergent. It is also seen, that if this equality is satisfied there will exist values of the phase u of the progressive disturbance for which v_1 is "supersonic" or "subsonic" with respect to v_L. The Appendix also shows that in the moving coordinate system solutions of the differential equation admit discontinuities in $\partial^2 E/\partial u^2$ across sonic lines, *i.e.*, lines of constant u for which $v_L = v_1$. On each side of such a sonic line a different solution will exist in general, neither of which can be analytically continued across the sonic line.

It is clear from the above discussion that Simon's harmonic expansion treatment of his triple-root case has no meaning in this context. We have not attempted in this paper to consider under what circumstances, if any, his treatment may be of value.

We have attempted here to enhance the value of Simon's interesting paper by presenting a rigorous solution and a sufficiency condition for the convergence of the solution, and by applying this condition to indicate that the treatment of Simon's triple-root case is in error.

Appendix

A. Convergence of the Space-Harmonic Expansion

We prove below that in the range

$$-1 < \frac{\epsilon_0}{\epsilon_1} D_\infty < 1,$$

where

$$D_\infty = \lim_{n\to\infty} D_n,$$

there exists for any real value of k a solution of the difference equation (6) for which the infinite determinant of the system vanishes, and for which the sum

$$\sum_{n=-\infty}^{\infty} | a_n |^2,$$

of the magnitude squared of the coefficients of the space-harmonic expansion diverges.

The existence of solutions of (6), consistent with the vanishing of the infinite determinant, for every real value of k can be demonstrated by a theorem of Weyl.[6] The possible existence of corresponding solutions with complex k was not investigated.

[6] F. Riesz and B. S. Nagy, "Functional Analysis," Frederick Ungar Co., New York, N. Y., p. 367; 1955.

The convergence of the above-mentioned sum can be determined from the asymptotic behavior of the coefficients a_n. To this end, let us write a general linear homogeneous difference equation of the nth order in the form

$$\sum_{i=0}^{n} p_i(x)u(x+i) = 0. \tag{27}$$

We shall assume that the coefficients $p_i(x)$ admit the following expansion, valid for sufficiently large values of x:

$$p_i(x) = p_i^{(0)} + p_i^{(1)}x^{-1}p_i^{(2)}x^{-2} + \cdots. \tag{28}$$

If now the so-called characteristic equation

$$p_n^{(0)}t^n + p_{n-1}^{(0)}t^{n-1} + \cdots + p_1^{(0)}t + p_0^{(0)} = 0 \tag{29}$$

possesses n different roots t_i, $1 \le i \le n$, then there exist n independent solutions of the difference equation (27) which are asymptotically represented in the form[7]

$$u_j(x) \sim (t_j)^x x^{r_j}\left(1 + \frac{b_{1j}}{x} + \frac{b_{2j}}{x^2} + \cdots\right), \quad 1 \le j \le n. \tag{30}$$

One determines the characteristic exponent r_j and the coefficients b_j by substitution of (30) into (27) and equating the sum of the coefficients of equal powers of x to zero.

Let us apply the above general result to the specific equation

$$a_{n+1} + \frac{2\epsilon_0}{\epsilon_1} D_n a_n + a_{n-1} = 0, \qquad n = 0, \pm 1, \cdots, \tag{31}$$

with D_n expressed in the form (valid for n sufficiently large)

$$D_n = D_\infty + \frac{d_1}{n} + \frac{d_2}{n^2} + \cdots. \tag{32}$$

The characteristic equation (29) now becomes

$$t^2 + \frac{2\epsilon_0}{\epsilon_1} D_\infty t + 1 = 0, \tag{33}$$

which has solutions

$$t_{1,2} = -e^{\pm j\theta}, \tag{34}$$

with $\cos\theta = (\epsilon_0/\epsilon_1)D_\infty$, as in (26). Hence, one finds that

$$a_{nj} \sim t_j^n n^{r_j}\left(1 + \frac{b_{1j}}{n} + \frac{b_{2j}}{n^2} + \cdots\right), \quad j = 1, 2. \tag{35}$$

[7] G. D. Birkhoff, "General theory of linear difference equations," *Trans. Am. Math. Soc.*, vol. 12, pp. 243–284; 1911.
J. Horn, "Zur Theorie der linearen Differenzengleichungen," *Math. Ann.*, vol. 53, pp. 177–192; 1900. Also, "Über das Verhalten der Integrale linearer Differenzen- und Differentialgleichungen für grosse Werte der Veränderlichen," *J. Reine angew. Math.*, vol. 138, pp. 159–191; 1910.

Substituting (35) into (31), we obtain

$$-t_j\left(1+\frac{1}{n}\right)^{r_j}\left(1+\frac{b_{1j}}{n\left(1+\frac{1}{n}\right)}+\cdots\right)$$

$$+\left(D_\infty+\frac{d_1}{n}+\cdots\right)\left(1+\frac{b_{1j}}{n}+\frac{b_{2j}}{n^2}+\cdots\right)$$

$$-(t_j)^{-1}\left(1-\frac{1}{n}\right)^{r_j}\left(1+\frac{b_{1j}}{n\left(1-\frac{1}{n}\right)}+\cdots\right)=0. \quad (36)$$

Equating to zero the coefficient of $(1/n)^0$, one has

$$\cos\theta=\frac{\epsilon_0}{\epsilon_1}D_\infty,$$

which agrees with (26) while equating to zero the coefficient of $(1/n)^1$, and using (34), one obtains the relation

$$r_{1,2}=\mp j\frac{d_1}{2\sin\theta}\cdot \quad (37)$$

Therefore, it follows from (35) that

$$(a_n)_{1,2}=(-1)^n e^{\pm jn\theta}e^{\mp jd_1 \ln n/2\sin\theta}\left[1+0\left(\frac{1}{n}\right)\right]. \quad (38)$$

In the range of parameters for which

$$-1<\frac{\epsilon_0}{\epsilon_1}D_\infty<1, \quad (39)$$

i.e., for which θ is real, and for d real, which corresponds to k real, as seen from (7) and (32), one finds

$$a_n-(-1)^n e^{\pm j\left[n\theta\mp(d_1\ln n/2\sin\theta)\right]}=0\left(\frac{1}{n}\right). \quad (40)$$

It is clear that under these conditions the sum

$$\sum_{n=-\infty}^{\infty}|a_n|^2$$

does not converge. This follows since the right-hand side of (40) is square-summable, and hence the left-hand side must be also. However, because the second term on the left-hand side is not square-summable, a_n cannot be. The divergence of this sum indicates the presence of a non-square-integrable singularity in the solution of the differential equation and, therefore, does not permit a harmonic expansion in this range.

B. Differential Equation Considerations and Sonic Lines

Further physical and mathematical insight into the singular nature of the solution in the range $(\epsilon_0/\epsilon_1)|D_\infty|\leq 1$ is afforded by considering the always hyperbolic differential equation

$$\frac{\partial^2 E(z,t)}{\partial z^2}-\mu_0\frac{\partial^2}{\partial t^2}\left[\epsilon(z,t)E(z,t)\right]=0, \quad (41)$$

where $\epsilon(z,t)$ is given by (1). Let us introduce the following transformation of variables appropriate to a moving coordinate system:

$$u=k_1 z-\omega_1 t;\qquad t'=t. \quad (42)$$

One obtains from (41) and (42) the transformed differential equation

$$(k_1^2-\mu_0\epsilon_0\omega_1^2-\mu_0\epsilon_1\omega_1^2\cos u)\frac{\partial^2 E(u,t')}{\partial u^2}$$

$$+2\omega_1(\mu_0\epsilon_0+\mu_0\epsilon_1\cos u)\frac{\partial^2 E}{\partial u\partial t'}-(\mu_0\epsilon_0+\mu_0\epsilon_1\cos u)\frac{\partial^2 E}{\partial t'^2}$$

$$+2\mu_0\epsilon_1\omega_1^2\sin u\frac{\partial E}{\partial u}-2\omega_1\mu_0\epsilon_1\sin u\frac{\partial E}{\partial t'}$$

$$-\mu_0\epsilon_1\omega_1^2\cos uE=0. \quad (43)$$

Upon recognizing that

$$v_1=\frac{\omega_1}{k_1}$$

and

$$v_0=\frac{\omega}{k_0}=\frac{1}{\sqrt{\mu_0\epsilon_0}}$$

are the phase velocities of the progressive disturbance (pump wave) and the electromagnetic wave in the unperturbed medium, respectively, and that the local phase velocity of the electromagnetic wave in the disturbed medium is

$$v_L=\frac{\omega}{k_0\sqrt{1+(\epsilon_1/\epsilon_0)\cos u}}, \quad (44)$$

(43) can be rewritten in the form

$$\left(1-\frac{v_1^2}{v_L^2(u)}\right)\frac{\partial^2 E}{\partial u^2}+\frac{2v_1}{k_1}\frac{1}{v_L^2(u)}\frac{\partial^2 E}{\partial u\partial t'}$$

$$-\frac{1}{k_1^2}\frac{1}{v_L^2(u)}\frac{\partial^2 E}{\partial t'^2}-2v_1^2\frac{d}{du}\left[\frac{1}{v_L^2(u)}\right]\frac{\partial E}{\partial u}$$

$$+2\frac{v_1}{k_1}\frac{d}{du}\left[\frac{1}{v_L^2(u)}\right]\frac{\partial E}{\partial t'}-v_1^2\left(\frac{1}{v_L^2(u)}-\frac{1}{v_0^2}\right)E=0. \quad (45)$$

In the range

$$\frac{\epsilon_0}{\epsilon_1}|D_\infty|\leq 1, \quad (46)$$

for which the sufficiency condition is not satisfied, there will always exist a real value $u=u_0$ for which the coefficient of $\partial^2E/\partial u^2$ in (43) vanishes, *i.e.*,

$$k_1^2 - \mu_0\epsilon_0\omega_1^2 - \mu_0\epsilon_1\omega_1^2 \cos u_0 = k_1^2\left(1 - \frac{v_1^2}{v_L^2(u_0)}\right) = 0. \quad (47)$$

The existence of real u_0 in this range is seen by rewriting (47) as

$$\cos u_0 = \frac{\epsilon_0}{\epsilon_1}\left[\frac{k_1^2/\omega_1^2}{\mu_0\epsilon_0} - 1\right]; \quad (48)$$

hence, u_0 is real if

$$\frac{\epsilon_0}{\epsilon_1}\left|\frac{(k_1/\omega_1)^2}{\mu_0\epsilon_0} - 1\right| \leq 1, \quad (49)$$

or

$$\frac{\epsilon_0}{\epsilon_1}\left| D_\infty \right| \leq 1,$$

which is identical with (46).

The vanishing of the coefficient of $\partial^2E/\partial u^2$, however, gives rise to a singularity of the differential equation (45). The singularity arises because even though the values of E and $\partial E/\partial u$ are given on the line $u = u_0$ for all t', the values of $\partial^2E/\partial u^2$ are not determined by the differential equation (45), and $\partial^2E/\partial u^2$ may jump across the line $u = u_0$. Hence, the "sonic lines," for which $v_1^2 = v_L^2(u_0)$, are singular lines of the differential equation (45). On each side of such a sonic line a different solution will exist, neither of which can be analytically continued across the sonic line. The sufficiency condition (17), therefore, eliminates the occurrence of such a singularity and guarantees the existence of square-integrable solutions of (45).

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A New Broad-Band Absorption Modulator for Rapid Switching of Microwave Power*

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Summary—This paper describes a new technique for obtaining a broad-band absorption modulator for high-speed switching or amplitude modulation of microwave power. This ferrite modulator, an outgrowth of the longitudinal-field rectangular-waveguide phase shifter,[1] has electrical characteristics particularly desirable in a microwave switch. These include a zero-field insertion loss of approximately 0.5 db in the ON state, an isolation of greater than 60 db in the OFF state which is nearly independent of the magnetic control field in this state, and a nearly matched input impedance for all values of applied field. These electrical characteristics are nearly constant over a 30 per cent bandwidth at X band. Also, it is possible to design the amplitude modulator to have negligible phase shift at the desired operating frequency.

Other characteristics of this ferrite modulator include small physical size, magnetic control fields of less than 50 oersteds, operating temperatures up to 150°C, and a capability of less than one μsec switching time.

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[1] F. Reggia and E. G. Spencer, "A new technique in ferrite phase-shifting for beam scanning of microwave antennas," PROC. IRE, vol. 45, pp. 1510–1517; November, 1957.

INTRODUCTION

In its most general form, the relationship between the induced RF flux density b and the internal RF magnetic field h in an arbitrarily magnetized polycrystalline-ferrite medium is a permeability tensor given by

$$\begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = \begin{bmatrix} \mu_{xx} & \mu_{xy} & \mu_{xz} \\ \mu_{yx} & \mu_{yy} & \mu_{yz} \\ \mu_{zx} & \mu_{zy} & \mu_{zz} \end{bmatrix} \begin{bmatrix} h_x \\ h_y \\ h_z \end{bmatrix}. \quad (1)$$

From this expression, Rado[2] has shown that for an unsaturated ferrite medium at microwave frequencies and a dc magnetic field applied in the z direction, the permeability tensor reduces to

$$[\mu] = \begin{bmatrix} \mu & -jK & 0 \\ +jK & \mu & 0 \\ 0 & 0 & \mu_z \end{bmatrix}, \quad (2)$$

[2] G. T. Rado, "Electromagnetic characterization of ferromagnetic media," IRE TRANS. ON ANTENNAS AND PROPAGATION, vol. AP-4, pp. 512–525; July, 1956.